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On the consistency problem for the $\mathcal{INDU}$ calculus ☆

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Abstract

In this paper, we further investigate the consistency problem for the qualitative temporal calculus $\mathcal{INDU}$ introduced by Pujari et al. [A.K. Pujari, G.V. Kumari, A. Sattar, INDU: An interval and duration network, in: Australian Joint Conference on Artificial Intelligence, 1999, pp. 291–303]. We prove the intractability of the consistency problem for the subset of pre-convex relations, and the tractability of strongly pre-convex relations. Furthermore, we also define another interesting set of relations for which the consistency problem can be decided by the $\diamond$-closure method, a method similar to the usual path-consistency method. Finally, we prove that the $\diamond$-closure method is also complete for the set of atomic relations of $\mathcal{INDU}$ implying that the intervals have the same duration.

Keywords: Qualitative temporal reasoning; Interval algebra; Qualitative constraint networks; Tractability

☆ This paper is an improved version of the paper [P. Balbiani, J.-F. Condotta, G. Ligozat, On the consistency problem for the INDU calculus, in: IEEE (Ed.), Proceeding of the Combined Tenth International Symposium on Temporal Representation and Reasoning and Fourth International Conference on Temporal Logic (TIME-ICTL 2003), Cairns, Queensland, Australia, 2003, pp. 203–211. [2]].

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1. Introduction

Temporal reasoning is a central task for numerous applications in many areas such as natural language understanding, specification and verification of programs and systems, scheduling, etc. In the field of qualitative reasoning about temporal data, the framework proposed by Allen [1], the Interval Algebra ($\mathcal{IA}$), is one of the best-known models.

Allen considers as basic temporal entities intervals of the time line and bases the $\mathcal{IA}$ calculus on 13 qualitative binary relations which correspond to all possible configurations between the four end-points of two intervals. In the $\mathcal{IA}$ calculus, temporal information can be represented using constraint networks (interval networks) whose variables correspond to intervals and whose constraints are expressed by disjunctions of the basic relations (interval relations). The consistency problem for interval networks is NP-complete. A large amount of research in the recent past has been devoted to the study and characterization of tractable subclasses of the interval algebra (see for example [4,11]). Now all tractable subclasses of $\mathcal{IA}$ are known.

More recently, a new qualitative formalism, called $\mathcal{INDU}$ has been proposed by Pujari et al. [7,8,12]. $\mathcal{INDU}$ also considers intervals as temporal entities, but it adds information about the relative durations of the intervals considered to the information expressed by Allen's relations. The resulting calculus has 25 basic relations corresponding to refinements of Allen's basic relations. Each one of the 25 basic relations of $\mathcal{INDU}$ can therefore be represented as a pair consisting of one of Allen's basic relations and of a basic relation of the Point Algebra ($<$, $>$ or $=$), which expresses the relation between the durations.

From a structural point of view, $\mathcal{INDU}$ and $\mathcal{IA}$ look very similar. This first impression, however, is quite deceptive. The real fact is that there exist numerous differences between the two formalisms. In particular, contrary to the relations of $\mathcal{IA}$, the relations of $\mathcal{INDU}$ are not closed for the composition operation. We also show that the consistency problem for $\mathcal{INDU}$ networks whose constraints are atomic relations cannot be decided by means of the well known path-consistency method.

In this paper, we are mainly interested in the consistency problem for $\mathcal{INDU}$ networks. Our aim is to characterize several important tractable sets for this problem. To this end we define the set of convex relations of $\mathcal{INDU}$ (in a way which is different from that used by Pujari et al.), the set of pre-convex relations, and a subset of the latter, the set of strongly pre-convex relations [3]. On the negative side, we prove that the consistency problem for $\mathcal{INDU}$ networks whose constraints are pre-convex relations is NP-complete. On the positive side, we show that strongly pre-convex relations can be expressed as conjunctions of Horn clauses [6] and, as a consequence, that the corresponding consistency problem is tractable. We also show that the usual method based on path-consistency cannot be used for strongly pre-convex relations. On the other hand, we define an interesting subclass of $\mathcal{INDU}$ relations for which the consistency problem can be decided by means of that method. Finally, we prove that the $\diamond$-closure method is also complete for the set of those atomic relations of $\mathcal{INDU}$ that imply that the intervals have the same duration.

2. The $\mathcal{INDU}$ calculus: an extension of $\mathcal{IA}$

2.1. The $\mathcal{INDU}$ relations

The framework introduced by Pujari et al. [12], called $\mathcal{INDU}$, is an extension of the well-known Allen's calculus [1], the Interval Algebra ($\mathcal{IA}$).

$\mathcal{IA}$ considers intervals of the line, and is based onto 13 binary relations. Each relation corresponds to a particular relative position between two intervals. An interval can be before (b), meet (m), overlap (o), contain (di), start (s), finish (f), be equal to (eq), another interval (the remaining six relations correspond to the converses of the first six relations). We denote by IA this set of 13 relations: $\mathsf{IA} = \{b, bi, m, mi, o, oi, s, si, f, fi, d, di, eq\}$. Vilain et al. [14] have defined a similar but less expressive formalism, the Point Algebra formalism ($\mathcal{PA}$). $\mathcal{PA}$ is based onto three basic relations $\mathsf{PA} = \{<, >, =\}$, corresponding to the three possible relative positions between two points of the line.

$\mathcal{INDU}$ is based on 25 binary relations between intervals of the line. Each one is defined as a pair of relations. The first relation is a basic relation of IA and denotes the relative position satisfied by the two intervals, the second one is a basic relation of PA and corresponds to the relation satisfied by the durations of the two intervals. Each basic relation of $\mathcal{INDU}$ will be denoted by an expression i^p where $i \in \mathsf{IA}$ and $p \in \mathsf{PA}$. Two intervals x and y satisfy i^p, denoted by $x\ i^p\ y$, iff $x\ i\ y$ and $(x^+ - x^-)\ p\ (y^+ - y^+)$, where x^-, y^- (resp. x^+, y^+) are the lower (resp. upper) endpoints of x and y. For example, the intervals[1] $(1, 2)$ and $(2, 4)$ satisfy the basic relation $m^<$. We denote by INDU the set of the 25 basic $\mathcal{INDU}$ relations, $\mathsf{INDU} = \{b^<, b^>, b^=, bi^<, bi^>, bi^=, m^<, m^>, m^=, mi^<, mi^>, mi^=, o^<, o^>, o^=, oi^<, oi^>, oi^=, s^<, si^>, d^<, di^>, f^<, fi^>, eq^=\}$. Any two intervals satisfy one and only one relation of INDU. In the sequel, we will also consider other basic relations which we call *virtual*. These particular basic relations are the unsatisfiable relations i^p. This set corresponds to $\{eq^<, eq^>, d^=, d^>, di^=, di^<, f^=, f^>, fi^=, fi^<, s^=, s^>, si^=, si^<\}$.

From the 25 basic relations of INDU, 2^{25} relations can be defined. Each one corresponds to an element of the power set of INDU, i.e., 2^{INDU}. Let $r \in 2^{\mathsf{INDU}}$, and x and y be two intervals, $x\ r\ y$ iff $x\ a\ y$ for some $a \in r$. For instance, $(1, 2)$ and $(3, 4)$ satisfy the relation $\{b^<, b^=, m^=, d^<\}$. The particular relation $\{\}$ is called the empty relation (we will also denote this relation by $\emptyset$). Each relation $r \in 2^{\mathsf{INDU}}$ can be seen as the disjunction of the basic relations which it contains. A relation containing only one basic relation is also called an atomic relation. Similarly, 2^{IA} defines 2^{13} interval relations and 2^{PA} defines 2^3 relations.

In some cases, the constraint "two intervals x and y must satisfy a relation $r \in \mathsf{INDU}$" can be expressed by a conjunction of Horn clauses. A Horn clause [6] is a disjunction of literals (inequations or inequalities) of the following forms: $c_1u_1 + \cdots + c_nu_n \leqslant c_{n+1}u_{n+1} + \cdots + c_{n+k}u_{n+k}$ or $c_1u_1 + \cdots + c_nu_n \neq c_{n+1}u_{n+1} + \cdots + c_{n+k}u_{n+k}$, with at most one literal using $\leqslant$ (each c_i is a rational number, each u_i is a rational variable). The basic relations belonging to INDU can be expressed by a conjunction of unitary Horn clauses (a unitary clause is a clause containing only one literal). For example, the satisfaction of

[1] We consider closed intervals of the rational number line. Such an interval will be denoted by (a, b) with a and b two rational numbers satisfying $a < b$.

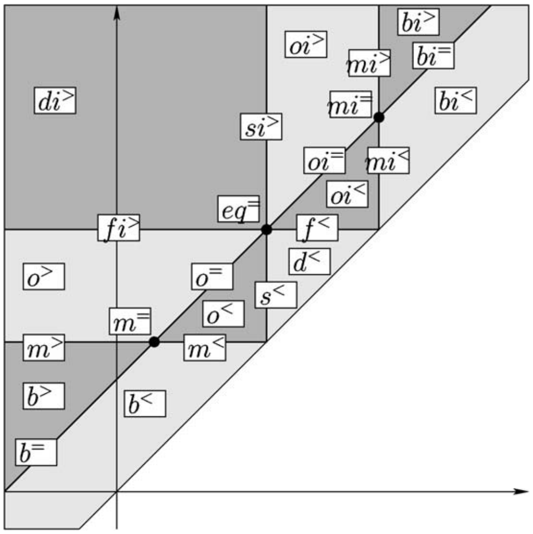

Fig. 1. Geometrical representation of the basic relations of $\mathcal{INDU}$.

$m^<$ by two intervals $x = (x^-, x^+)$ and $y = (y^-, y^+)$ can be expressed by the following conjunction of unitary Horn clauses: $(x^- \leqslant x^+) \wedge (x^- \neq x^+) \wedge (y^- \leqslant y^+) \wedge (y^- \neq y^+) \wedge (x^+ \leqslant y^-) \wedge (y^- \leqslant x^+) \wedge (x^+ - x^- \leqslant y^+ - y^-) \wedge (x^+ - x^- \neq y^+ - y^-)$. A ORD Horn clause is a Horn clause which contains only literals of the forms $u \neq v$ or $u \leqslant v$, with u, v two endpoints. All relations of 2^{PA} can be expressed by a conjunction of ORD Horn clauses, those of 2^{IA} having this property are called the ORD Horn relations [11].

In the same way as for the basic relations of IA, the basic relations of $\mathcal{INDU}$ can be represented by regions of the plane equipped with an orthogonal basis. Each interval $x = (x^-, x^+)$ is represented by a point with coordinates (x^-, x^+). Given an interval of reference $y = (y^-, y^+)$, the region representing the basic relation $a \in \mathsf{INDU}$, denoted by $\mathsf{Reg}(a, y)$, corresponds to the set of points $\{x = (x^-, x^+)\colon\ x\ a\ y\}$. We obtain 25 convex regions partitioning the half plane $\mathsf{H} = \{(x, y)\colon\ y > x\}$ (see Fig. 1). The geometrical representation of $r \in 2^{\mathsf{INDU}}$ is the region defined by $\mathsf{Reg}(r, y) = \bigcup_{a \in r}\{\mathsf{Reg}(a, y)\}$. When the choice of the point of reference y is immaterial, we denote this region by $\mathsf{Reg}(r)$.

2.2. The fundamental operations

Similarly to the case of relations of $\mathcal{IA}$ and $\mathcal{PA}$, we define operations for the relations of $\mathcal{INDU}$. Considering a relation of 2^{INDU} as a usual binary relation defined on intervals, the operations union ($\cup$), intersection ($\cap$), converse ($^{-1}$) and composition ($\circ$) can be defined in the usual way: $x(r \cap s)y$ iff $x\ r\ y$ and $x\ s\ y$; $x(r \cup s)y$ iff $x\ r\ y$ or $x\ s\ y$; $x(r \circ s)y$ iff $\exists z$, $x\ r\ z$ and $z\ s\ y$; $x\ r^{-1}\ y$ iff $y\ r\ x$. It is easy to show that $r \cap s = \{a \in \mathsf{INDU}\colon\ a \in r \text{ and } a \in s\}$, $r \cup s = \{a \in \mathsf{INDU}\colon\ a \in r \text{ or } a \in s\}$. The converse of an atomic relation is an atomic relation, like for IA and PA, and $\{i^p\}^{-1} = \{(i^{-1})^{p^{-1}}\}$, with $i^p \in \mathsf{INDU}$. Hence, $r^{-1} = \bigcup_{a \in r}\{a^{-1}\}$. Hence, 2^{INDU} is closed for $\cap$, $\cup$ and $^{-1}$. The $\mathcal{INDU}$ composition operation has an unusual behavior for a qualitative formalism. Indeed, unlike $\mathcal{IA}$ and $\mathcal{PA}$, 2^{INDU} is not closed under composition: consider the relation $r = \{m^=\}$, the pair of intervals formed by $(1,2)$ and $(3,4)$ belongs to $r \circ r$ as $(1,2)$ and $(2,3)$ satisfy $m^=$, $(2,3)$ and $(3,4)$ satisfy $m^=$. Since $(1,2)$ and $(3,4)$ satisfy the relation $b^=$, $b^= \in r \circ r$. Now, given a pair of intervals (x, y) satisfying the basic relation $b^=$, an interval z such that $x\ m^=\ z$ and $z\ m^=\ y$ may not exist. For example, this is the case for $x = (0,1)$ and $y = (4,5)$. So, the composition operation is inadequate for qualitative reasoning in $\mathcal{INDU}$ since *basic building blocks* must be the basic relations. It is necessary to define a weaker operation, sometimes called *weak composition*, for which 2^{INDU} is closed; we will denote this operation by $\diamond$. The operation $\diamond$ is defined as follows: Firstly, for atomic relations $a, b \in \mathsf{INDU}$, $a \diamond b = \{c \in \mathsf{INDU}$: $\exists x, y, z$ with $x\ a\ z, z\ b\ y$ and $x\ c\ y\}$; then, for arbitrary relations $r, s \in 2^{\mathsf{INDU}}$, $r \diamond s = \bigcup_{a \in r, b \in s} \{a \diamond b\}$. Equivalently, $r \diamond s$ is the smallest relation of 2^{INDU} containing $r \circ s$. Note that $\circ$ and $\diamond$ are the same operations for $\mathcal{IA}$ and $\mathcal{PA}$. The operation $\diamond$ is not associative, for instance, $(\{bi^>\} \diamond \{mi^>\}) \diamond \{m^>\} = \{oi^>, mi^>, bi^>\}$ and $\{bi^>\} \diamond (\{mi^>\} \diamond \{m^>\}) = \{bi^>\}$. As a result $\diamond$ cannot be used to define a relation algebra [13] on INDU. We also define a binary operation corresponding to the Cartesian product of an interval relation and a point relation by: $r \times s = \{i^p\colon\ i \in r, p \in s\}$, with $r \in 2^{\mathsf{IA}}$ and $s \in 2^{\mathsf{PA}}$. This relation can contain virtual basic relations of $\mathcal{INDU}$. Note that for $i^p, j^q \in \mathsf{INDU}$, $i^p \diamond j^q = ((i \circ j) \times (p \circ q)) \cap \mathsf{INDU}$. The interval and point projections of an $\mathcal{INDU}$ relation r, denoted respectively, by r_I and r_P are defined by $r_I = \{i\colon\ i^p \in r\}$, $r_P = \{p\colon\ i^p \in r\}$. In the sequel we will say that a subset of relations of 2^{INDU} is a subclass iff it is closed for the operations ${}^{-1}$, $\diamond$, and $\cap$.

3. Qualitative constraint networks

3.1. Definitions

Definition 1. An $\mathcal{INDU}$ constraint network is a pair $\mathcal{N} = (V, C)$, where:

- V is a finite set $\{V_1, \ldots, V_n\}$ (with $n = |V|$) of variables representing intervals of the line;
- C is a mapping associating with each pair $V_i, V_j \in V$ a constraint, denoted by C_{ij}, defined by a relation of 2^{INDU}.

We assume that $C_{ij}^{-1} = C_{ji}$ and $C_{ii} = \{eq^=\}$.

We define constraint networks on $\mathcal{IA}$ (interval networks) and on $\mathcal{PA}$ (point networks) in a similar way. By definition, an atomic network is a network whose constraints are defined by atomic relations.

Definition 2. Let $\mathcal{N} = (V, C)$ be a constraint network on $\mathcal{INDU}$ with $n = |V|$. An instantiation m of $\mathcal{N}$ is a mapping which associates an interval m_i with each variable $V_i \in V$. The basic relation of INDU satisfied by m_i and m_j will be denoted by m_{ij}. The instantiation m will be called a consistent instantiation or a solution of $\mathcal{N}$ iff for every pair of variables $V_i, V_j \in V$, $m_{ij} \in C_{ij}$. In the case where $\mathcal{N}$ has a solution, $\mathcal{N}$ will be said consistent. $\mathcal{N}$ is $k-$consistent (with $k \in \{1, \ldots, n\}$) iff any partial consistent instantiation on $k-1$ variables can be extended to a new variable while remaining consistent. $\mathcal{N}$ will be said

$\diamond$-closed iff for each $i, j, k \in \{1, \ldots, n\}$, $C_{ij} \subseteq C_{ik} \diamond C_{kj}$. A subnetwork $\mathcal{N}' = (V', C')$ is a network such that $V' = V$ and $C'_{ij} \subseteq C_{ij}$ for each pair of variables V_i and V_j. A network $\mathcal{N}'' = (V, C'')$ is equivalent to $\mathcal{N}$ iff $\mathcal{N}$ and $\mathcal{N}''$ have the same solutions.

Given a constraint network, the main problem is to decide whether it admits a consistent instantiation. This problem is called the consistency problem.

Given a set $\mathcal{E} \subseteq 2^{\mathsf{INDU}}$ (closed for converse and containing the relation $\{eq^{=}\}$), the consistency problem for $\mathcal{INDU}$ networks whose constraints belong to $\mathcal{E}$ will be denoted by $\mathsf{Cons}(\mathcal{E})$. The consistency problem for interval networks being NP-complete, obviously $\mathsf{Cons}(2^{\mathsf{INDU}})$ is NP-hard. Moreover, we will show in Section 4.2 that the consistency problem for atomic $\mathcal{INDU}$ networks is a polynomial problem. Hence, we can test the consistency of any $\mathcal{INDU}$ network in exponential time by testing the consistency of all its atomic subnetworks. As a consequence, we get the fact that $\mathsf{Cons}(2^{\mathsf{INDU}})$ is NP-complete.

We call $\diamond$-closure method, the method which consists in obtaining from a network $\mathcal{N} = (V, C)$ an equivalent and $\diamond$-closed subnetwork $\mathcal{N}'$ by iterating the operation $C_{ij} := C_{ij} \cap (C_{ik} \diamond C_{kj})$, for $i, j, k \in \{1, \ldots, |V|\}$, until a fixpoint is obtained. This method can be implemented in $\mathrm{O}(n^3)$ time (with $n = |V|$) by an algorithm similar to those used to obtain equivalent path-consistent constraint subnetworks from binary constraint networks [10].

3.2. Consistency and the $\diamond$-closure method

Given that the set 2^{INDU} is not closed for the composition operation, several fundamental properties of networks of $\mathcal{IA}$ and $\mathcal{PA}$ are no longer true in the framework of $\mathcal{INDU}$.

Proposition 1. *Let $\mathcal{N}$ be a consistent $\mathcal{INDU}$ network. A* 3*-consistent network $\mathcal{N}'$ equivalent to $\mathcal{N}$ may not exist.*

The atomic $\mathcal{INDU}$ network depicted in Fig. 2(a) is $\diamond$-closed and consistent but it is not 3-consistent: consider the partial solution $m(V_1) = (1, 2)$, $m(V_3) = (3, 5)$. This solution cannot be extended to V_2. This network is consistent and does not admit an equivalent 3-consistent network.

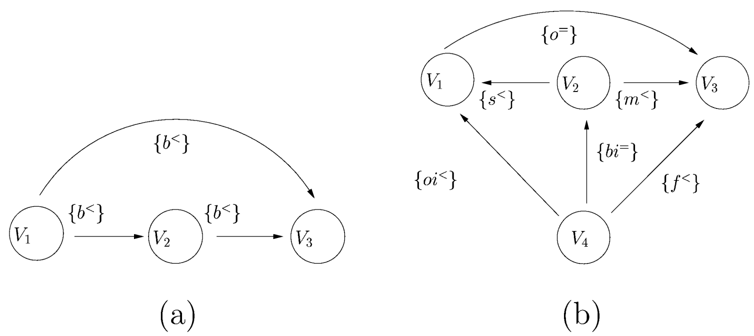

Fig. 2. (a) a consistent atomic network which is not 3-consistent, (b) an atomic network which is $\diamond$-closed but not consistent.

Moreover, there are $\diamond$-closed atomic networks which are not consistent (for example, see Fig. 2(b)).

Consequently, we can state the following property:

Proposition 2. *A $\diamond$-closed (atomic) $\mathcal{INDU}$ network which does not contain the empty constraint is not necessarily consistent.*

For $\mathcal{IA}$ it has been shown that the complexity of $\mathsf{Cons}(\mathcal{E})$, with $\mathcal{E} \in 2^{\mathsf{IA}}$, is the same as the complexity of $\mathsf{Cons}(\overline{\mathcal{E}})$, with $\overline{\mathcal{E}}$ the closure of $\mathcal{E}$ for the operations converse, intersection and composition. The proof of this result is based on the fact that from an interval network whose constraints belong to $\overline{\mathcal{E}}$ we can always build an equivalent network from $\mathcal{E}$. If we replace the composition operation by $\diamond$, we can no longer prove this property for $\mathcal{INDU}$. This is a consequence of the fact that if $x\ (r \diamond s)\ y$ then a third interval z such that $x\ r\ z$ and $z\ s\ y$ may not exist (a necessary property for building an equivalent network on $\mathcal{E}$ from the network on $\overline{\mathcal{E}}$). Nevertheless, we can prove a weaker property:

Proposition 3. *Let $\mathcal{E} \subseteq 2^{\mathsf{INDU}}$ be a subset of relations which is closed under the converse operation and which contains the atomic relation $\{eq^=\}$. Then $\mathsf{Cons}(\mathcal{E})$ is a polynomial problem (resp. a NP-complete problem) if, and only if, $\mathsf{Cons}(\mathcal{E}^*)$ is a polynomial problem (resp. a NP-complete problem), where * denotes the closure for the intersection operation.*

A proof of this proposition can be based on the fact that a constraint $x\ (r_0 \cap \cdots \cap r_i)\ y$ in $\mathcal{E}^*$ (with $r_0, \ldots, r_i \in \mathcal{E}$) can be replaced by the constraints $x\ r_0\ y$, $x\ r_1\ z_1, \ldots, x\ r_i\ z_i$, $y\ \{eq^=\}\ z_1, \ldots, y\ \{eq^=\}\ z_i$, where $z_1, \ldots, z_i$ are new variables. Hence, we have a polynomial reduction from $\mathsf{Cons}(\mathcal{E}^*)$ to $\mathsf{Cons}(\mathcal{E})$.

In the sequel of this paper, we are going to characterize several sets of $\mathcal{INDU}$ for which the consistency problem is polynomial. Several cases of tractability can be obtained in a direct way from the tractable cases of $\mathcal{IA}$. For instance, we can state the following result:

Proposition 4. *Let $\mathcal{E} \subseteq 2^{\mathsf{IA}}$ be a set for which the consistency problem is polynomial. Let $\mathcal{E}' \subseteq 2^{\mathsf{INDU}}$ be defined by $\mathcal{E}' = \{(r \times \{<, =, >\}) \cap \mathsf{INDU}\colon\ r \in \mathcal{E}\}$.*

Then $\mathsf{Cons}(\mathcal{E}')$ is polynomial.

The validity of this proposition is due to the fact that $\mathcal{E}$ and $\mathcal{E}'$ represent the same class of temporal constraints. Indeed, the $\mathcal{INDU}$ temporal constraint $x\ ((r \times \{<, =, >\}) \cap \mathsf{INDU})\ y$ (with x, y two variables and $r \in \mathcal{E}$) can be equivalently expressed by the temporal constraint of the interval algebra $x\ r\ y$. Let us now establish less trivial tractability cases.

4. Convex relations in the $\mathcal{INDU}$ calculus

4.1. Definition and representation

In this section, we introduce a lattice structure on the set of basic $\mathcal{INDU}$ relations, based on the similar structures for $\mathcal{IA}$ and $\mathcal{PA}$ [9] (see also [12]). In [9] the interval lattice

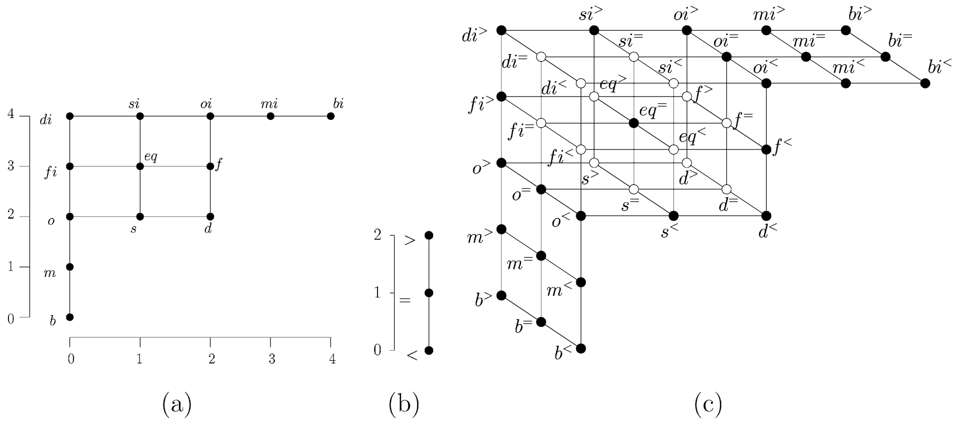

Fig. 3. (a) the interval lattice, (b) the point lattice, (c) the $\mathcal{INDU}$ lattice.

(resp. the point lattice) is defined by associating to each basic relation A a pair of integers (i_A, j_A) (resp. an integer i_A). For example, the pair $(1, 3)$ corresponds to the basic relation eq, while 1 corresponds to the basic relation $=$ (see Fig. 3 for a complete description). Using this correspondence, an ordering $\leqslant_{int}$ (resp. $\leqslant_{pt}$) is defined on IA (resp. on PA) by specifying that $A \leqslant_{int} B$ iff $i_A \leqslant i_B$ and $j_A \leqslant j_B$ (resp. $i_A \leqslant i_B$). We refer to the resulting structure $(\mathsf{IA}, \leqslant_{int})$ (resp. $(\mathsf{PA}, \leqslant_{pt})$) as to the interval lattice (resp. the point lattice).

Once the lattice has been defined, convex relations of $\mathcal{IA}$ (resp. $\mathcal{PA}$) correspond to intervals in the lattice[2] (resp. the point lattice) (see Fig. 3 (a) and (b)). For example, the interval relation $\{o, s, fi, eq\}$ and the point relation $\{<, =\}$ are convex relations corresponding to the intervals $[o, eq]$ and $[<, =]$.

In a natural way, we define the $\mathcal{INDU}$ lattice as the Cartesian product of the interval lattice and the point lattice (see Fig. 3). This lattice is also defined on the virtual basic relations of $\mathcal{INDU}$. We define the set of convex relations of 2^{INDU} in the following way:

Definition 3. A relation $r \in 2^{\mathsf{INDU}}$ is a convex relation iff $r = [\min, \max] \cap \mathsf{INDU}$, where $[\min, \max]$ an interval in the $\mathcal{INDU}$ lattice.

For example, the relation $\{m^<, m^=, o^<, o^=\}$ is a convex relation. We denote by $\mathcal{C}$ the set of convex relations. Remark that $r \in 2^{\mathsf{INDU}}$ is convex iff $r = (s \times t) \cap \mathsf{INDU}$ with s and t convex relations of 2^{IA} and 2^{PA}. Hence from a geometrical point of view, a relation r of $\mathcal{INDU}$ is a convex relation when its geometrical representation in the plane $\mathsf{Reg}(r)$ satisfies the following equality: $\exists h \in \mathcal{H}$, $\mathsf{Reg}(r) = (Proj_1(\mathsf{Reg}(r)) \times Proj_2(\mathsf{Reg}(r))) \cap h$, where $\mathcal{H} = \{\mathsf{Reg}(\mathsf{INDU}), \mathsf{Reg}(\mathsf{INDU} \cap (\mathsf{IA} \times \{<\})), \mathsf{Reg}(\mathsf{INDU} \cap (\mathsf{IA} \times \{<, =\})), \mathsf{Reg}(\mathsf{INDU} \cap (\mathsf{IA} \times \{>\})), \mathsf{Reg}(\mathsf{INDU} \cap (\mathsf{IA} \times \{>, =\})), \mathsf{Reg}(\mathsf{INDU} \cap (\mathsf{IA} \times \{=\}))\}$ and where $Proj_1$ (resp. $Proj_2$) denote the projection functions on the horizontal axis (resp. vertical axis).

[2] Given a lattice $(E, \leqslant)$, an interval is either the empty set or a set $\{e \in E\colon \min \leqslant e \leqslant \max\}$ for some $\min, \max \in E$ with $\min \leqslant \max$ (this last set will be denoted by $[\min, \max]$).

In the same way as for the convex relations of $\mathcal{IA}$ and of $\mathcal{PA}$ we have the following property for the convex relation of $\mathcal{INDU}$:

Proposition 5. *A relation $r \in \mathsf{INDU}$ is a convex relation iff it can be expressed by a conjunction of unitary Horn clauses Φ such that if $u \neq v \in \Phi$ then $u \leqslant v \in \Phi$ or $v \leqslant u \in \Phi$ (where u and v denote endpoints or differences of endpoints).*

Proof. Let $r \in \mathsf{INDU}$ be a convex relation. We know that $r = (s \times t) \cap \mathsf{INDU}$ where s and t are convex relations in 2^{IA} and 2^{PA} respectively. Now s is a convex relation of 2^{IA} which expresses the constraint concerning the relative position between the two intervals. This constraint can be expressed by a conjunction of unitary Horn clauses Φ_s such that if $u \neq v \in \Phi_s$ then $u \leqslant v \in \Phi_s$ or $v \leqslant u \in \Phi_s$ (with u and v denoting endpoints). In a similar way, the duration constraint between the two intervals is expressed by t a convex relation of 2^{PA}. Again, t can be also expressed by a conjunction of unitary Horn clauses Φ_t such that if $u \neq v \in \Phi_t$ then $u \leqslant v \in \Phi_t$ or $v \leqslant u \in \Phi_t$ (with u and v denoting differences of endpoints). Let us remark that u and v denote differences of endpoints in Φ_s and endpoints in Φ_t. Finally, we can define the required conjunction Φ by $\Phi_s \wedge \Phi_t$. $\square$

For example, consider the $\mathcal{INDU}$ convex relation $r = \{m^=, m^<, o^=, o^<\}$. We have $r = (s \times t) \cap \mathsf{INDU}$ with $s = \{m, o\}$ and $t = \{<, =\}$. The constraint $x\ r\ y$ can be expressed by $\Phi_s = y^- \leqslant x^+ \wedge x^+ \leqslant y^+ \wedge x^- \leqslant x^+ \wedge y^- \leqslant y^+ \wedge x^+ \neq y^+ \wedge x^- \neq x^+ \wedge y^- \neq y^+$. The duration constraint t can be expressed by $\Phi_t = x^+ - x^- \leqslant y^+ - y^-$. Hence, the temporal constraint $x\ r\ y$ can be expressed by the conjunction $\Phi_s \wedge \Phi_t$.

Notice that a convex relation of $\mathcal{INDU}$ cannot always be represented by a conjunction of unitary ORD Horn clauses (see the previous example). Pujari et al. enumerate 227 convex relations. Our definition, by contrast, results in 240 convex relations, this difference arising from the fact that Pujari et al. use a lattice which does not take into account the virtual basic relations. The set $\mathcal{C}$ is closed for $^{-1}$, $\cap$, but not for $\diamond$. The closure for the intersection and converse follows directly from the definition. The following example shows the instability of $\mathcal{C}$ for $\diamond$: $\{b^<\} \diamond \{d^<, o^<, o^>, o^=, s^<\}$ is the non-convex relation $\{b^<, b^>, b^=, d^<, o^<, m^<, s^<\}$. Some relations of $\mathcal{C}$ can be expressed by conjunctions of ORD Horn clauses. This set of relations, denoted by $\mathcal{C}_{IA}$, corresponds to the 83 convex relations of $\mathcal{IA}$. $\mathcal{C}_{IA}$ can be defined as follows:

Definition 4. Let r be a relation in 2^{INDU}. Then $r \in \mathcal{C}_{IA}$ iff r satisfies one of the equivalent properties:

- $r = (s \times \{<, =, >\}) \cap \mathsf{INDU}$ where s is a convex interval relation,
- $r = [a^<, b^>] \cap \mathsf{INDU}$, where $[a^<, b^>]$ is an interval of the $\mathcal{INDU}$ lattice ($a, b \in \mathsf{IA}$).

Given that the set $\mathcal{C}_{IA}$ corresponds to the set of convex interval relations, the operation $\diamond$ on $\mathcal{C}_{IA}$ corresponds to the composition operation $\circ$. As a further consequence, $\mathcal{C}_{IA}$ is a subclass.

4.2. Tractability of the convex $\mathcal{INDU}$ relations

The convex $\mathcal{INDU}$ relations can be represented by conjunctions of unitary Horn clauses; as a consequence, the consistency problem of the convex $\mathcal{INDU}$ networks ($\mathcal{INDU}$ networks whose constraints are convex) is polynomial. Indeed, we can translate this kind of network into conjunctions of Horn clauses and apply a resolution algorithm such as the algorithm proposed by Koubarakis [6]. Notice that we can also use the Simplex algorithm or Kachian's algorithm for solving these particular constraints (as a consequence of Proposition 5).

Proposition 6. $\mathsf{Cons}(\mathcal{C})$ *is a polynomial problem.*

It is well known that the path-consistency method can be used to solve the consistency problem of the convex interval networks. Hence, $\mathsf{Cons}(\mathcal{C}_{IA})$ can be decided by the $\diamond$-closure method.

5. The pre-convex $\mathcal{INDU}$ relations

The maximal tractable set of $\mathcal{IA}$ containing the 13 atomic interval relations is the set of pre-convex interval relations, which is identical with the set of ORD Horn interval relations. To define the pre-convex relations of $\mathcal{INDU}$ we use the method introduced by Ligozat [9] by extending the notions of convex closure and dimension to $\mathcal{INDU}$. For the interval algebra, the dimension of an interval relation corresponds to the dimension of the geometrical representation of this region in the plane. This dimension is the maximal dimension of the dimensions of the basic relations it contains. In a similar way, we have the following definition:

Definition 5. Let $a \in \mathsf{INDU}$. The dimension of a, denoted by $\dim(a)$, is the dimensional space of $\mathsf{Reg}(a)$. If $r \in 2^{\mathsf{INDU}}$ is a non-empty relation, $\dim(r) = \max\{\dim(a)\colon a \in r\}$.

As usual, we define $\dim(\{\})$ as -1. As an illustration, we have $\dim(m^=) = 0$, $\dim(s^<) = 1$, $\dim(b^>) = 2$ and $\dim(\{m^=, o^<\}) = \max\{0, 2\} = 2$.

Definition 6. The convex closure of an $\mathcal{INDU}$ relation r, denoted by $I(r)$, is the smallest convex relation of $\mathcal{C}$ containing r: $I(r) = \bigcap\{s \in \mathcal{C}\colon r \subseteq s\}$.

Notice that this definition makes sense because the set $\mathcal{C}$ is closed under intersection.

The convex closure in $\mathcal{INDU}$ can be computed from the convex closures in $\mathcal{IA}$ and $\mathcal{PA}$. Indeed, we have $I(r) = (I(r_I) \times I(r_P)) \cap \mathsf{INDU}$ for all relations $r \in 2^{\mathsf{INDU}}$. Now, we can define the pre-convex relations of $\mathcal{INDU}$:

Definition 7. Let $r \in 2^{\mathsf{INDU}}$. Then r is a pre-convex relation iff $r = \{\}$ or $\dim(I(r) \setminus r) < \dim(r)$.

We denote by $\mathcal{P}$ the set of pre-convex $\mathcal{INDU}$ relations. $\mathcal{P}$ contains 88096 relations. The set of convex relations $\mathcal{C}$ is a subset of $\mathcal{P}$. The set $\mathcal{P}$ is closed under $^{-1}$, but not closed under the operations $\cap$ and $\diamond$: consider the pre-convex $\mathcal{INDU}$ relations $r = \{eq^=, b^<, b^=, o^<\}$, $s = \{eq^=, b^>, b^=, o^>\}$, $t = \{b^<\}$ and $u = \{d^<, o^<, o^>\}$. Then the relations $r \cap s = \{eq^=, b^=\}$ and $t \diamond u = \{b^<, b^>, b^=, d^<, o^<, m^<, s^<\}$ are not pre-convex relations.

5.1. Intractability of the pre-convex $\mathcal{INDU}$ relations

In this section we prove that the consistency problem for $\mathcal{P}$ is NP-complete. In order to do so, we define a polynomial reduction from the 3-coloring problem of a graph [5] to $\mathsf{Cons}(\mathcal{P}^*)$.

Proposition 7. $\mathsf{Cons}(\mathcal{P}^*)$ *is a NP-complete problem.*

Proof. Let $G = (S, A)$ be a non-oriented graph, with S a set of vertices and A a set of edges between these vertices. We build an $\mathcal{INDU}$ network $\mathcal{N} = (V, C)$ in the following way: V is a set of variables corresponding to the union of $Col = \{Col_1, Col_2, Col_3\}$ and $V_S = \{S_1, \ldots, S_n\}$ with $n = |S|$. Each variable of Col is associated with a color. Each variable $S_i \in V_S$ is associated with a vertex $s_i \in S$. The constraints of $\mathcal{N}$ between the three variables of Col are given in Fig. 4(a). Those between two variables S_i and S_j such that $(s_i, s_j) \in A$ (resp. $\notin A$) are given in Fig. 4(b) (resp. (c)). We can check that these constraints belong to $\mathcal{P}^*$. For example, the relation $\{m^=, eq^=, mi^=\}$ is the intersection of the pre-convex relations $\{o^>, di^>, oi^>, m^=, eq^=, mi^=\}$ and $\{o^<, d^<, o^<, m^=, eq^=, mi^=\}$. We can prove that $G = (S, A)$ is 3-colorable iff $\mathcal{N}$ is consistent: given a solution of the 3-coloring problem for G, we assign to each variable S_i the interval corresponding to the color assigned to the vertex s_i. Conversely, to obtain a solution of the 3-coloring problem for G from a solution of $\mathcal{N}$, we assign to the vertex s_i the color corresponding to the interval assigned to S_i. $\square$

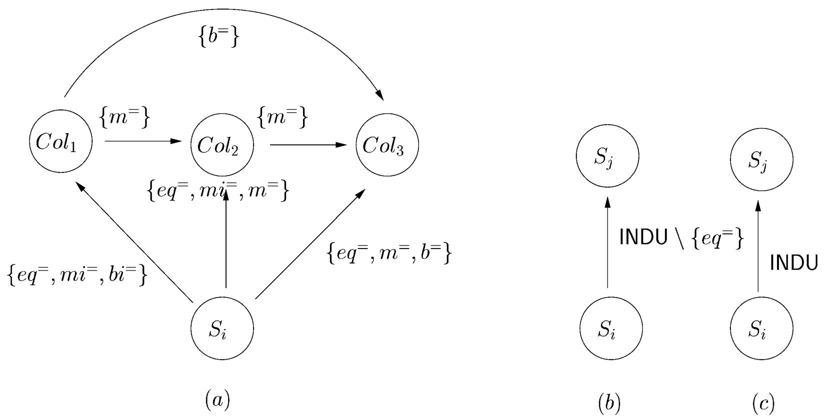

Fig. 4. The constraints of $\mathcal{N} = (V, C)$.

As a consequence of this proposition and Proposition 3 we obtain:

Theorem 1. $\mathsf{Cons}(\mathcal{P})$ *is a NP-complete problem.*

6. The strongly pre-convex $\mathcal{INDU}$ relations

Balbiani et al. have proved that for the strongly pre-convex generalized interval relations the consistency problem is polynomial [3]. Following the line of reasoning given by Balbiani et al., we define the strongly pre-convex relations of $\mathcal{INDU}$.

Definition 8. Let $r \in 2^{\mathsf{INDU}}$. Then r is strongly pre-convex iff for each convex relation $t \in \mathcal{C}$, the relation $r \cap t$ is a pre-convex relation.

The definition of the strongly pre-convex relations is guided by the desire to obtain a subset of pre-convex relations closed under the intersection operation. $\mathcal{F}$ will denote the set of strongly pre-convex relations of $\mathcal{INDU}$. It has 45792 elements.

Proposition 8. *The set $\mathcal{F}$ is closed*[3] *under the operations $^{-1}$, $\cap$, but not closed under $\diamond$.*

As a counter-example, consider the strongly pre-convex relations $r = \{b^<\}$ and $s = \{d^<, o^<, o^>\}$, $r \diamond s$ is the relation $\{b^<, b^>, b^=, d^<, o^<, m^<, s^<\}$ which is not a strongly pre-convex relation.

6.1. Tractability of the strongly pre-convex $\mathcal{INDU}$ relations

This section is devoted to the properties of the strongly pre-convex $\mathcal{INDU}$ relations in relation to the consistency problems.

Proposition 9. *The strongly pre-convex $\mathcal{INDU}$ relations can be represented by conjunctions of Horn clauses.*

Proof. Let $r \in \mathcal{F}$. As $I(r)$ is convex, there exists a conjunction of Horn clauses representing it. Denote by $\Phi_{I(r)}$ such a conjunction. In the general case, $\Phi_{I(r)}$ is too permissive. Indeed, each basic relation $a \in I(r) \setminus r$ is realizable w.r.t. $\Phi_{I(r)}$. We must forbid these basic relations, without forbidding the basic relations of r. For each such $a \in I(r) \setminus r$, we exhibit a Horn clause, denoted by Φ_a, such that the addition of Φ_a to Φ excludes the satisfaction of a without excluding that of the atomic relations belonging to r. Since r is pre-convex, then $\dim(I(r) \setminus r) < \dim(r)$, hence $\dim(a) < \dim(r)$. Consequently, $\dim(a) = 1$ or 0. Let us first consider the basic relations which do not impose equality on the durations.

$$\Phi_{m^<} = (x^+ \neq y^- \vee x^+ - x^- \geqslant y^+ - y^-),$$

[3] A computer-program has been used to prove this result, as well as for the future Proposition 10.

$$\Phi_{m^>} = (x^+ \neq y^- \vee x^+ - x^- \leqslant y^+ - y^-),$$
$$\Phi_{mi^<} = (y^+ \neq x^- \vee x^+ - x^- \geqslant y^+ - y^-),$$
$$\Phi_{mi^>} = (y^+ \neq x^- \vee x^+ - x^- \leqslant y^+ - y^-),$$
$$\Phi_{s^<} = (x^- \neq y^- \vee x^+ - x^- \geqslant y^+ - y^-),$$
$$\Phi_{si^>} = (x^- \neq y^- \vee x^+ - x^- \leqslant y^+ - y^-),$$
$$\Phi_{f^<} = (x^+ \neq y^+ \vee x^+ - x^- \geqslant y^+ - y^-),$$
$$\Phi_{fi^>} = (x^+ \neq y^+ \vee x^+ - x^- \leqslant y^+ - y^-).$$

Next, consider the basic relations imposing equality of the durations. These atomic relations belong to the convex relation $s = \{eq^=, b^=, bi^=, o^=, oi^=, m^=, mi^=\}$. As a consequence, Φ_a will always contain $x^+ - x^- \neq y^+ - y^-$ (except for Φ_{eq}).

$$\Phi_{b^=} = (x^+ - x^- \neq y^+ - y^- \vee x^+ \geqslant y^-),$$
$$\Phi_{bi^=} = (x^+ - x^- \neq y^+ - y^- \vee y^+ \geqslant x^-),$$
$$\Phi_{m^=} = (x^+ - x^- \neq y^+ - y^- \vee x^+ \neq y^-),$$
$$\Phi_{mi^=} = (x^+ - x^- \neq y^+ - y^- \vee y^+ \neq x^-),$$
$$\Phi_{eq^=} = (x^- \neq y^- \vee x^+ \neq y^+).$$

The cases of the basic relations $o^=$ and $oi^=$ remain to be examined. Consider the case $a = o^=$ (the case $oi^=$ is similar). Suppose that $r \cap \{b^=, m^=\} \neq \emptyset$ and $r \cap \{eq^=, mi^=, oi^=, bi^=\} \neq \emptyset$. Hence $a \in I(r \cap s)$. Moreover, we know that $a \notin r$. As a consequence $\dim(I(r \cap s) \setminus (r \cap s)) \geqslant 1$. Since $r \cap s \subseteq s$ and $\dim(s) = 1$, $\dim(I(r \cap s) \setminus (r \cap s)) \leqslant 1$. Hence, $\dim(r \cap s) \leqslant \dim(I(r \cap s) \setminus (r \cap s))$ and $r \cap s$ is not pre-convex. This is a contradiction (r is a strongly pre-convex relation). Hence, only three cases have to be considered:

- $r \cap s = \emptyset$. Then $\Phi_{o^=}$ is $x^+ - x^- \neq y^+ - y^-$,
- $r \cap \{b^=, m^=\} \neq \emptyset$ and $r \cap \{eq^=, mi^=, oi^=, bi^=\} = \emptyset$. Then $\Phi_{o^=}$ is $x^+ - x^- \neq y^+ - y^- \vee x^+ \leqslant y^-$,
- $r \cap \{b^=, m^=\} = \emptyset$ and $r \cap \{eq^=, mi^=, oi^=, bi^=\} \neq \emptyset$. Then $\Phi_{o^=}$ is $x^+ - x^- \neq y^+ - y^- \vee x^+ \geqslant y^+$.

Using the clauses defined above, any $r \in \mathcal{F}$ can be represented by the conjunction of Horn clauses $\Phi_{I(r)} \wedge \bigwedge_{a \in (I(r) \setminus r)} \Phi_a$. □

As a consequence, we get:

Theorem 2. $\mathsf{Cons}(\mathcal{F})$ *is a polynomial problem.*

7. The tractable subclass $\mathcal{G}$

In this section we characterize a new subset of pre-convex relations for which the $\diamond$-closure method gives a decision method for the consistency problem (contrarily to what is the case for $\mathcal{F}$). We will denote this set by $\mathcal{G}$. The definition of $\mathcal{G}$ was guided by our desire to obtain pre-convex relations forming a subclass for which the convex closures are convex interval relations.

Definition 9. Let $r \in 2^{\mathsf{INDU}}$. Then r belongs to $\mathcal{G}$ iff for each convex relation $s \in \mathcal{C}_{IA}$ $r \cap s$ is a pre-convex relation and $I(r \cap s)$ is a convex relation which belongs to the set $\mathcal{C}_{IA}$.

The set $\mathcal{G}$ forms a subclass containing 11854 relations.

Proposition 10. *The set $\mathcal{G}$ is closed for the operations $^{-1}$, $\cap$ and $\diamond$.*

Since the universal relation INDU (that is, the set $INDU$) belongs to $\mathcal{C}_{IA}$, each relation of $\mathcal{G}$ is a pre-convex relation. Moreover, we notice that some relations of $\mathcal{G}$ are not strongly pre-convex. For example, the relation $\{eq^=, d^<, di^>, o^<, o^>, oi^<, oi^>, m^<, m^>, m^=, mi^<, mi^>, mi^=\}$ belongs to $\mathcal{G}$ but is not strongly pre-convex: indeed its intersection with the convex relation $\{eq^=, o^=, oi^=, m^=, mi^=\}$ is not a pre-convex relation.

7.1. Tractability of $\mathcal{G}$

We are now in a position to prove the tractability of the consistency problem for the set $\mathcal{G}$, using the notion of maximal solution introduced by Ligozat in [9]. Given a solution m of a network $\mathcal{N} = (V, C)$, m will be said maximal if $\dim(m_{ij}) = \dim(C_{ij})$ for all $i, j \in 1, \ldots, |V|$. Intuitively, a maximal solution is a solution which involves basic relations imposing as few equalities—between endpoints and difference of endpoints—as possible. For example, given the constraint $x\ \{m^=, m^>, o^=, o^<, b^>\}\ y$, a maximal solution will satisfy $o^<$ or $b^>$ between x and y. Firstly, we prove the following result:

Proposition 11. *Let $\mathcal{N} = (V, C)$ a $\mathcal{INDU}$ network whose constraints belong to $\mathcal{C}_{IA}$ (with the exclusion of the empty relation). If $\mathcal{N}$ is $\diamond$-closed then $\mathcal{N}$ admits a maximal solution.*

Proof. $\mathcal{C}_{IA}$ corresponds to the convex relations of $\mathcal{IA}$. Let $\mathcal{N}' = (V, C')$ be the convex interval network equivalent to $\mathcal{N}$. From [9] we know that $\mathcal{N}'$ admits a solution $m_1, \ldots, m_n$ (with $n = |V|$) such that $a = b$, with a and b two endpoints of m_i and m_j, iff all basic relations belonging to C'_{ij} impose this equality (m is a maximal solution for $\mathcal{IA}$). For example, we have $m_i^- = m_j^-$ iff all basic relations of C'_{ij} impose this equality. C'_{ij} could be the relation $\{s, eq, si\}$ but could not be the relation $\{s, eq, f, d\}$ since d does not impose the equality $m_i^- = m_j^-$. We can modify m to obtain a solution s having the additional property: $s_i^+ - s_i^- = s_j^+ - s_j^-$ iff $C_{ij} = \{eq^=\}$. Consider the lower endpoint m_i^-, let l be the number of endpoints located before m_i^-. We assign to s_i^- the value $l/(1+l)$. We treat in a similar way the upper endpoints. s satisfies the properties fixed previously. Hence, we obtain a maximal solution s of $\mathcal{N} = (V, C)$. $\square$

Proposition 12. *Let $r, s \in$ INDU such that $I(r \diamond s)$, $I(r)$ and $I(s) \in \mathcal{C}_{IA}$. We have $I(r \diamond s) \subseteq I(r) \diamond I(s)$.*

Proof. $r \subseteq I(r)$ and $s \subseteq I(s)$. Hence $r \diamond s \subseteq I(r) \diamond I(s)$. Consequently, $I(r \diamond s) \subseteq I(I(r) \diamond I(s))$. As $\mathcal{C}_{IA}$ is closed for the operation $\diamond$, $I(r) \diamond I(s)$ is a convex relation. Hence, $I(I(r) \diamond I(s)) = I(r) \diamond I(s)$. Hence, $I(r \diamond s) \subseteq I(r) \diamond I(s)$. □

Proposition 13. *Let $\mathcal{N} = (V, C)$ be a network whose constraints belong to $\mathcal{G}$. Let $\mathcal{N}^I = (V, C^I)$ be defined by $C^I_{ij} = I(C_{ij})$ for all $i, j \in \{1, \dots, n\}$, with $n = |V|$. If $\mathcal{N}$ is $\diamond$-closed then $\mathcal{N}^I$ is $\diamond$-closed.*

Proof. Let $V_i, V_j, V_k \in V$. $C_{ij} \subseteq C_{ik} \diamond C_{kj}$, consequently, $I(C_{ij}) \subseteq I(C_{ik} \diamond C_{kj})$. We know that $\mathcal{G}$ is closed for the operation $\diamond$. Hence, $I(C_{ik} \diamond C_{kj}) \in \mathcal{C}_{IA}$. Moreover, by definition of $\mathcal{G}$, $I(C_{ik})$ and $I(C_{kj}) \in \mathcal{C}_{IA}$. From Proposition 12, we get that $I(C_{ik} \diamond C_{kj}) \subseteq I(C_{ik}) \diamond I(C_{kj})$. Using this result, we deduce that $I(C_{ij}) \subseteq I(C_{ik}) \diamond I(C_{kj})$. □

Now, we can establish the main result concerning the set $\mathcal{G}$.

Theorem 3. $\mathsf{Cons}(\mathcal{G})$ *can be decided by means of the $\diamond$-closure method.*

Proof. Let $\mathcal{N} = (V, C)$ be a network containing constraints belonging to $\mathcal{G}$. By using the $\diamond$-closure method on $\mathcal{N}$ we obtain an equivalent subnetwork $\mathcal{N}' = (V, C')$. The constraints of $\mathcal{N}'$ belong to $\mathcal{G}$ since $\mathcal{G}$ is closed for the three operations $^{-1}$, $\cap$ and $\diamond$. If $\mathcal{N}'$ contains the empty constraint, then $\mathcal{N}$ is not consistent. In the opposite case, let us show that $\mathcal{N}'$ (and consequently $\mathcal{N}$) is consistent. Let $\mathcal{N}'' = (V, C'')$ be defined by $C''_{ij} = I(C'_{ij})$. $\mathcal{N}''$ is $\diamond$-closed (Proposition 13). It admits a maximal solution m (Proposition 11). This solution m is also a maximal solution of $\mathcal{N}'$. This is due to the fact that $\dim(I(C'_{ij}) \setminus C'_{ij}) < \dim(C'_{ij})$ (see definition of $\mathcal{G}$), for each pair of variables V_i and V_j. □

Hence, we have characterized a set for which the $\diamond$-closure method is complete.

8. The atomic relations of $\mathsf{INDU}^=$

In the previous section we showed that the $\diamond$-closure solves the consistency problem $\mathsf{Cons}(\mathcal{G})$.[4] On the other hand, we know that for the general case, the $\diamond$-closure method is not complete for the consistency problem of the atomic networks of $\mathcal{INDU}$. In this section, we show that this method is complete for the atomic relations which imply the equality of the durations of the intervals. In the sequel, we will denote by $\mathsf{INDU}^=$ the subset of the basic relations of $\mathcal{INDU}$ implying the equality between the durations of two intervals, that is $\mathsf{INDU}^= = \{eq^=, b^=, bi^=, m^=, mi^=, o^=, oi^=\}$. Notice that the atomic relations defined from $\mathsf{INDU}^=$, excepted $\{eq^=\}$, are convex relations of $\mathcal{INDU}$ which do not belong to the set $\mathcal{C}_{AI}$.

Given an $\mathcal{INDU}$ constraint network $\mathcal{N} = (V, C)$, we will denote by $\mathcal{N}^{\mathsf{IA}}$ the interval constraint network (V, C^{IA}) defined as follows: for each $i, j \in 1, \dots, |V|$, $C^{\mathsf{IA}}_{ij} = \{i\colon i^p \in$

[4] Note that the only basic $\mathcal{INDU}$ relations belonging to $\mathcal{G}$ are $\{eq^=\}$, $\{d^<\}$, $\{di^>\}$, $\{f^<\}$, $\{fi^>\}$, $\{s^<\}$, $\{si^>\}$.

$C_{ij}\}$. Similarly, we will denote by $\mathcal{N}^{\mathsf{PA}}$ the point constraint network (V, C^{PA}) defined as follows: for each $i, j \in 1, \ldots, |V|$, $C^{\mathsf{PA}}_{ij} = \{p\colon\ i^p \in C_{ij}\}$. The constraint networks $\mathcal{N}^{\mathsf{IA}}$ and $\mathcal{N}^{\mathsf{PA}}$ are, respectively, the projection of $\mathcal{N}$ onto the Interval Algebra and the projection of $\mathcal{N}$ onto the Point Algebra. In the general case, it is clear that the consistency of $\mathcal{N}^{\mathsf{IA}}$ and the one of $\mathcal{N}^{\mathsf{PA}}$ do not imply the consistency of $\mathcal{N}$.

The projection operation retains the property of $\diamond$-closure:

Proposition 14. *Let $\mathcal{N} = (V, C)$ an $\mathcal{INDU}$ constraint network. If $\mathcal{N}$ is a $\diamond$-closed network then $\mathcal{N}^{\mathsf{IA}}$ and $\mathcal{N}^{\mathsf{PA}}$ are $\diamond$-closed networks* (*and also $\circ$-closed networks*).

Proof. Denote by n the number of elements of the set V. Let $i, j, k \in 1, \ldots, n$. Let $a \in C^{\mathsf{IA}}_{ij}$ (resp. $b \in C^{\mathsf{PA}}_{ij}$), there exists $b \in \mathsf{PA}$ (resp. $a \in \mathsf{IA}$) such that $a^b \in C_{ij}$. By definition of the operation $\diamond$ and by $\diamond$-closure of $\mathcal{N}$, for each basic relation $a^b \in C_{ij}$, there exist $c^d \in C_{ik}$ and $e^f \in C_{kj}$ such that $a^b \in (c^d \diamond e^f)$. Moreover, $c^d \diamond e^f = ((c \circ e) \times (d \circ f)) \cap \mathsf{INDU}$. Hence $a \in (c \circ e)$ and $b \in (d \circ f)$. By projection, $c \in C^{\mathsf{IA}}_{ik}$, $e \in C^{\mathsf{IA}}_{kj}$, $d \in C^{\mathsf{PA}}_{ik}$ and $f \in C^{\mathsf{PA}}_{kj}$. Consequently, $a \in (C^{\mathsf{IA}}_{ik} \circ C^{\mathsf{IA}}_{kj})$ and $b \in (C^{\mathsf{PA}}_{ik} \circ C^{\mathsf{PA}}_{kj})$. Hence $C^{\mathsf{IA}}_{ij} \subseteq (C^{\mathsf{IA}}_{ik} \circ C^{\mathsf{IA}}_{kj})$ and $C^{\mathsf{PA}}_{ij} \subseteq (C^{\mathsf{PA}}_{ik} \circ C^{\mathsf{PA}}_{kj})$. Hence, $\mathcal{N}^{\mathsf{IA}}$ and $\mathcal{N}^{\mathsf{PA}}$ are $\circ$-closed networks. For the Interval Algebra and the Point Algebra, the operations $\circ$ and $\diamond$ are the same operations, consequently, $\mathcal{N}^{\mathsf{IA}}$ and $\mathcal{N}^{\mathsf{PA}}$ are $\diamond$-closed networks too. □

Now, we are going to prove that some particular interval constraint networks admit solutions enforcing a common duration for all intervals.

Proposition 15. *Let $\mathcal{N} = (V, C)$ an atomic network of the Interval Algebra such that each constraint is formed by one basic relation belonging to the set $S = \{eq, m, mi, b, bi, o, oi\}$. If $\mathcal{N}$ is a $\circ$-closed network* (*and does not have the empty relation as a constraint*) *then $\mathcal{N}$ admits a consistent instantiation σ such that $\sigma(V_i)^+ - \sigma(V_i)^- = \sigma(V_j)^+ - \sigma(V_j)^-$ for all $V_i, V_j \in V$.*

Proof. Let $\mathcal{N} = (V, C)$ be an atomic interval constraint network such that $C_{ij} = \{A\}$ with $A \in \{eq, m, mi, b, bi, o, oi\}$. Suppose that $\mathcal{N}$ is a $\circ$-closed network. As $\mathcal{N}$ is $\circ$-closed and atomic we know that there exists a consistent instantiation σ' of $\mathcal{N}$ [11]. From this instantiation, we build a new consistent instantiation, denoted by σ, using uniquely intervals with same duration, as follows: Without loss of generality, we suppose that the variables $V = V_1, \ldots, V_n$ are such that if $\sigma'^{+}_i < \sigma'^{+}_j$ then $i < j$, for all $i, j \in 1, \ldots, n$. Thus, if $i < j$, then, $C_{ij} \subseteq \{eq, m, b, o\}$. In the sequel of this proof we will denote by T the interval relation $\{eq, m, b, o\}$ and d will denote the difference $\sigma'^{+}_1 - \sigma'^{-}_1$. Moreover, we define the sets E^A_l, with $A \in \{eq, b, m, o, oi, mi, bi\}$ and $l \in 1, \ldots, n$, by $E^A_l = \{i \in 1, \ldots, l-1\colon\ C_{il} = \{A\}\}$. Let the instantiation σ defined in the following way:

- $\sigma_1 = \sigma'_1$,
- for each $k \in 2, \ldots, n$,
  - if $C_{k-1k} = \{eq\}$ then $\sigma_k = \sigma_{k-1}$,
  - if $C_{k-1k} = \{m\}$ then $\sigma_k = (\sigma^+_{k-1}, \sigma^+_{k-1} + d)$,

– if $C_{k-1k} = \{b\}$ then $\sigma_k = (\sigma^+_{k-1} + d, \sigma^+_{k-1} + 2d)$,
– if $C_{k-1k} = \{o\}$ then
  if $E^m_k \neq \emptyset$ then $\sigma_k = (\sigma^+_i, \sigma^+_i + d)$, with $i = \min\ E^m_k$,
  else $\sigma_k = ((u + v)/2, (u + v)/2 + d)$, with $v = \sigma^+_{\min\ E^o_k}$ and $u = \sigma^+_{\max(E^b_k \cap E^o_{k-1})}$ in the case where $(E^b_k \cap E^o_{k-1}) \neq \emptyset$ else $u = \sigma^-_{k-1}$.

Let us prove that the instantiation σ satisfies the properties required. Firstly, we can remark that for all $k \in 1, \ldots, n$, $\sigma^+_k - \sigma^-_k = d$. Hence, all the intervals used in the instantiation σ have the same duration. Now, we show that σ is a consistent instantiation of $\mathcal{N} = (V, C)$. Let P be the property defined by $P(k)$ (with $k \in 1, \ldots, n$) be satisfied if, and only if, the partial instantiation $\sigma_1, \ldots, \sigma_k$ is a consistent instantiation of the network $\mathcal{N}$. $P(1)$ is trivially true. Let $k \in 2, \ldots, n$, suppose that $P(k-1)$ is true, we show that the property P is satisfied for k. We know that the constraint C_{k-1k} is a atomic relation such that $C_{k-1k} \subseteq T$. Examine all possible cases concerning the constraint C_{k-1k}.

- $C_{k-1k} = eq$. We have $\sigma_k = \sigma_{k-1}$. Hence, $\sigma_{k-1}\ C_{k-1k}\ \sigma_k$. Moreover, for each $l \in 1, \ldots, k-2$, from the $\circ$-closure of $\mathcal{N}$, we have $C_{lk} \subseteq C_{lk-1} \circ C_{k-1k}$. Moreover, $C_{lk-1} \circ C_{k-1k} = C_{lk-1} \circ \{eq\} = C_{lk-1}$. Since the constraints of $\mathcal{N}$ are atomic relations, $C_{lk} = C_{lk-1}$. As $P(k-1)$ is true, $\sigma_l\ C_{lk-1}\ \sigma_{k-1}$. From the fact that $\sigma_{k-1} = \sigma_k$ and $C_{lk-1} = C_{lk}$, we deduce that $\sigma_l\ C_{lk}\ \sigma_k$.
- $C_{k-1k} = m$. Hence, $\sigma_k = (\sigma^+_{k-1}, \sigma^+_{k-1} + d)$. Consequently, we have $\sigma_{k-1}\ C_{k-1k}\ \sigma_k$. From the $\circ$-closure of $\mathcal{N}$, we have $C_{lk} \subseteq C_{lk-1} \circ C_{k-1k}$ for each $l \in 1, \ldots, k-2$. Moreover, $C_{lk-1} \subseteq \{eq, m, b, o\}$ and $C_{k-1k} = \{m\}$. Since $\{eq, m, b, o\} \circ \{m\} = \{m, b\}$, we obtain the inclusion $C_{lk} \subseteq \{m, b\}$. Consider the two possible cases for C_{lk}:
  - $C_{lk} = \{b\}$. From the $\circ$-closure of $\mathcal{N}$, we have $C_{lk-1} \subseteq C_{lk} \circ C_{kk-1}$. Hence, $C_{lk-1} \subseteq \{b\} \circ \{mi\}$. Since $\{b\} \circ \{mi\} = \{b, m, o, d, s\}$ and $C_{lk-1} \subseteq T$ we can assert that $C_{lk-1} \subseteq \{b, m, o\}$. As $P(k-1)$ is satisfied, we have $\sigma_l\ C_{lk-1}\ \sigma_{k-1}$. Consequently, $\sigma^+_l < \sigma^+_{k-1}$. Moreover, remember that $\sigma^+_{k-1} = \sigma^-_k$. We deduce that $\sigma^+_l < \sigma^-_k$. We can conclude that $\sigma_l\ \{b\}\ \sigma_k$ and hence, $\sigma_l\ C_{lk}\ \sigma_k$.
  - $C_{lk} = \{m\}$. From the $\circ$-closure of $\mathcal{N}$, we have $C_{lk-1} \subseteq C_{lk} \circ C_{kk-1}$. Hence, $C_{lk-1} \subseteq \{m\} \circ \{mi\}$. From the fact that $\{m\} \circ \{mi\} = \{f, fi, eq\}$ and $C_{lk-1} \subseteq T$ we get that $C_{lk-1} = \{eq\}$. Since $P(k-1)$ is satisfied, we have $\sigma_l\ C_{lk-1}\ \sigma_{k-1}$. Consequently, $\sigma_l = \sigma_{k-1}$. Moreover, remember that $\sigma^+_{k-1} = \sigma^-_k$. We conclude that $\sigma_l\ \{m\}\ \sigma_k$ and hence, $\sigma_l\ C_{lk}\ \sigma_k$.
- $C_{k-1k} = b$. Hence $\sigma_k = (\sigma^+_{k-1} + d, \sigma^+_{k-1} + 2d)$. Consequently, $\sigma_{k-1}\ C_{k-1k}\ \sigma_k$. From the $\circ$-closure of $\mathcal{N}$, for each $l \in 1, \ldots, k-2$, we get $C_{lk} \subseteq C_{lk-1} \circ C_{k-1k}$. Hence, $C_{lk} \subseteq T \circ \{b\}$. From the fact that $\{eq, m, b, o\} \circ \{b\} = \{b\}$ we can deduce that $C_{lk} = \{b\}$. As $P(k-1)$ is true, $\sigma_l\ C_{lk-1}\ \sigma_{k-1}$. From the fact that $C_{lk-1} \subseteq \{eq, m, b, o\}$, we get $\sigma^+_l \leqslant \sigma^+_{k-1}$. Moreover, notice that $\sigma^+_{k-1} < \sigma^-_k$. Hence, we can assert that $\sigma^+_l < \sigma^-_k$. Consequently, $\sigma_l\ \{b\}\ \sigma_k$ and hence, $\sigma_l\ C_{lk}\ \sigma_k$.
- $C_{k-1k} = o$. We must take into account two possible cases: the case where $E^m_k \neq \emptyset$ and the case where $E^m_k = \emptyset$.

- $E_k^m \neq \emptyset$. Let $i = \min E_k^m$ (notice that $1 \leqslant i < k$ and $C_{ik} = \{m\}$). We have $\sigma_k = (\sigma_i^+, \sigma_i^+ + d)$. Let $l \in 1, \ldots, k-1$. We know that $C_{lk} \in T$. Examine all possible cases concerning C_{lk}.
  - * $C_{lk} = \{eq\}$. From the $\circ$-closure of $\mathcal{N}$, we deduce that $C_{il} \subseteq C_{ik} \circ C_{kl}$. Hence, $C_{il} \subseteq \{m\} \circ \{eq\}$. Moreover, $\{m\} \circ \{eq\} = \{m\}$. Consequently, $C_{il} = \{m\}$. Since $P(k-1)$ is satisfied, $\sigma_i m \sigma_l$. Hence, $\sigma_l^- = \sigma_i^+$. As the duration of σ_l is d, $\sigma_l = (\sigma_i^+, \sigma_i^+ + d)$. Consequently, $\sigma_l = \sigma_k$ and hence, $\sigma_l\ \{eq\}\ \sigma_k$. We deduce that $\sigma_l\ C_{lk}\ \sigma_k$.
  - * $C_{lk} = \{m\}$. From the $\circ$-closure of $\mathcal{N}$, $C_{il} \subseteq C_{ik} \circ C_{kl}$. Consequently, $C_{il} \subseteq \{m\} \circ \{mi\}$. From the fact $\{m\} \circ \{mi\} = \{f, fi, eq\}$ and $C_{il} \subseteq S$, we can assert that $C_{il} = \{eq\}$. Since $P(k-1)$, $\sigma_i\ \{eq\}\ \sigma_l$. Hence, $\sigma_l = \sigma_i$. As $\sigma_k^- = \sigma_i^+$, we have $\sigma_k^- = \sigma_l^+$. Consequently, $\sigma_l\ \{m\}\ \sigma_k$ and hence, $\sigma_l\ C_{lk}\ \sigma_k$.
  - * $C_{lk} = \{b\}$. From the $\circ$-closure of $\mathcal{N}$, we can deduce that $C_{il} \subseteq C_{ik} \circ C_{kl}$. Consequently, $C_{il} \subseteq \{m\} \circ \{bi\}$. Moreover, $\{m\} \circ \{bi\} = \{bi, oi, mi, di, si\}$ and, we know that $C_{il} \subseteq S$. Hence, $C_{il} \subseteq \{bi, oi, mi\}$ and $C_{li} \subseteq \{b, o, m\}$. Since the property $P(k-1)$ is true, we have $\sigma_l\ C_{li}\ \sigma_i$. Consequently, we can assert that $\sigma_l^+ < \sigma_i^+$. As $\sigma_k^- = \sigma_i^+$, we get that $\sigma_k^- > \sigma_l^+$. Consequently, $\sigma_l\, b\, \sigma_k$.
  - * $C_{lk} = \{o\}$. From the $\circ$-closure of $\mathcal{N}$, $C_{il} \subseteq C_{ik} \circ C_{kl}$. Consequently, $C_{il} \subseteq \{m\} \circ \{oi\}$. $\{m\} \circ \{oi\} = \{o, d, s\}$ and moreover, we know that $C_{il} \subseteq S$. Hence, $C_{il} = \{o\}$. As the property $P(k-1)$ is true, we have $\sigma_i\ C_{il}\ \sigma_l$. Consequently, we can assert that $\sigma_i^- < \sigma_l^- < \sigma_i^+ < \sigma_l^+$, with $\sigma_l^+ - \sigma_i^+ < d$. Moreover, we know that $\sigma_k = (\sigma_i^+, \sigma_i^+ + d)$. Hence, $\sigma_l^- < \sigma_k^- < \sigma_l^+ < \sigma_k^+$. Consequently, $\sigma_l\ \{o\}\ \sigma_k$. Hence, $\sigma_l\ C_{lk}\ \sigma_k$.
- $E_k^m = \emptyset$. Denote by i the element corresponding to $\min E_k^o$. Notice that i exists since $C_{k-1k} = \{o\}$. Two cases must be considered: the case where $E_k^b \cap E_{k-1}^o = \emptyset$ and the case where $E_k^b \cap E_{k-1}^o \neq \emptyset$.
  - * $E_k^b \cap E_{k-1}^o = \emptyset$. Hence, $\sigma_k = ((\sigma_i^+ + \sigma_{k-1}^-)/2, (\sigma_i^+ + \sigma_{k-1}^-)/2 + d)$. From the $\circ$-closure of $\mathcal{N}$, we can deduce that $C_{ik-1} \subseteq C_{ik} \circ C_{kk-1}$. Consequently, $C_{ik-1} \subseteq \{o\} \circ \{oi\}$. From the fact that $\{o\} \circ \{oi\} = \{o, oi, d, s, f, di, si, fi, eq\}$ and $C_{ik-1} \subseteq T$ (since $i \leqslant k-1$). We deduce that $C_{ik-1} \subseteq \{o, eq\}$. Consequently,we have $\sigma_i^- \leqslant \sigma_{k-1}^- < \sigma_i^+ \leqslant \sigma_{k-1}^+$, with $\sigma_{k-1}^+ - \sigma_{k-1}^- = \sigma_i^+ - \sigma_i^- = d$. As $\sigma_k = ((\sigma_i^+ + \sigma_{k-1}^-)/2, (\sigma_i^+ + \sigma_{k-1}^-)/2 + d)$ we can assert that $\sigma_i^- \leqslant \sigma_{k-1}^- < \sigma_k^- < \sigma_i^+ \leqslant \sigma_{k-1}^+ < \sigma_k^+$. Let $l \in 1, \ldots, k-1$. Consider all possible cases concerning the constraint C_{lk}. We know that $C_{lk} \subseteq T$, hence, we must consider four cases.
    - – $C_{lk} = \{eq\}$. From the $\circ$-closure of $\mathcal{N}$, we can deduce that $C_{lk-1} \subseteq C_{lk} \circ C_{kk-1}$. Consequently, $C_{lk-1} \subseteq \{eq\} \circ \{oi\}$, hence, $C_{lk-1} = \{oi\}$. Consequently, ${\sigma'_l}^+ > {\sigma'_{k-1}}^+$. This implies that $l > k-1$. This is a contradiction.
    - – $C_{lk} = \{m\}$. By hypothesis, $E_k^m = \emptyset$. This is a contradiction.
    - – $C_{lk} = \{b\}$. From the $\circ$-closure of $\mathcal{N}$, $C_{lk-1} \subseteq C_{lk} \circ C_{kk-1}$. We deduce that $C_{lk-1} \subseteq \{b\} \circ \{oi\}$. Moreover, $\{b\} \circ \{oi\} = \{b, o, m, d, s\}$ and we know that $C_{lk} \subseteq T$. Consequently, $C_{lk-1} \subseteq \{b, o, m\}$. In the case where $C_{lk-1} \subseteq \{b, m\}$, from the fact that $P(k-1)$ is true, we have $\sigma_l\ \{b, m\}\ \sigma_{k-1}$. Hence, $\sigma_l^+ \leqslant \sigma_{k-1}^-$. As $\sigma_k^- > \sigma_{k-1}^-$, we have $\sigma_l^+ < \sigma_k^-$. Hence, we can assert that $\sigma_l\ \{b\}\ \sigma_k$ and

hence, $\sigma_l\ C_{lk}\ \sigma_k$. Now, suppose that $C_{lk-1} = \{o\}$ (notice that in this case $l < k-1$). Hence, $E_k^b \cap E_{k-1}^o \neq \emptyset$. This is a contradiction.

– $C_{lk} = \{o\}$. By definition of i we have $i \leqslant l$. Hence, $i \leqslant l \leqslant k-1$. Consequently, $C_{il} \subseteq T$ and $C_{lk-1} \subseteq T$. As the property $P(k-1)$ is satisfied, we have $\sigma_i\ C_{il}\ \sigma_l$ and $\sigma_l\ C_{lk-1}\ \sigma_{k-1}$. Hence, $\sigma_i^- \leqslant \sigma_l^- \leqslant \sigma_{k-1}^-$ and $\sigma_i^+ \leqslant \sigma_l^+ \leqslant \sigma_{k-1}^+$. Moreover, remember us that $\sigma_i^- \leqslant \sigma_{k-1}^- < \sigma_k^- < \sigma_i^+ \leqslant \sigma_{k-1}^+ < \sigma_k^+$. Consequently, we have $\sigma_i^- \leqslant \sigma_l^- \leqslant \sigma_{k-1}^- < \sigma_k^- < \sigma_i^+ \leqslant \sigma_l^+ \leqslant \sigma_{k-1}^+ < \sigma_k^+$. We deduce that $\sigma_l\ \{o\}\ \sigma_k$. Hence, $\sigma_l\ C_{lk}\ \sigma_k$.

* $E_k^b \cap E_{k-1}^o \neq \emptyset$. We deduce that $\sigma_k = ((\sigma_i^+ + \sigma_j^+)/2, (\sigma_i^+ + \sigma_j^+)/2 + d)$, with $j = \max(E_k^b \cap E_{k-1}^o)$. As in the previous case we have $C_{ik-1} \subseteq \{o, eq\}$ and hence $\sigma_i^- \leqslant \sigma_{k-1}^- < \sigma_i^+ \leqslant \sigma_{k-1}^+$ with $\sigma_{k-1}^+ - \sigma_{k-1}^- = \sigma_i^+ - \sigma_i^- = d$. Moreover, from the $\circ$-closure of $\mathcal{N}$, we deduce that $C_{ij} \subseteq C_{ik-1} \circ C_{k-1j}$. We have $C_{k-1j} = \{oi\}$ since $j \in E_{k-1}^o$. Hence, $C_{ij} \subseteq \{o, eq\} \circ \{oi\}$. Moreover, $\{o, eq\} \circ \{oi\} = \{eq, o, oi, s, si, f, fi, d, di\}$. We know that $C_{ij} \subseteq S$. Consequently, we can assert that $C_{ij} \subseteq \{eq, o, oi\}$. Similarly, from the $\circ$-closure of $\mathcal{N}$, we have $C_{ij} \subseteq C_{ik} \circ C_{kj}$. $C_{kj} = \{bi\}$ since $j \in E_k^b$ and $C_{ik} = \{o\}$ since $i \in E_k^o$. Consequently, $C_{ij} \subseteq \{o\} \circ \{bi\}$. Moreover, $\{o\} \circ \{bi\} = \{bi, oi, di, mi, si\}$ and $C_{ij} \subseteq S$. Hence, $C_{ij} \subseteq \{bi, oi, mi\}$. It results that $C_{ij} \subseteq \{eq, o, oi\} \cap \{bi, oi, mi\}$. Hence, we have $C_{ij} = \{oi\}$. As $P(k-1)$ is true, we have $\sigma_i\ C_{ij}\ \sigma_j$. It follows that $\sigma_j^- < \sigma_i^- < \sigma_j^+ < \sigma_i^+$. Moreover, $\sigma_j^- < \sigma_{k-1}^- < \sigma_j^+ < \sigma_{k-1}^+$ since $C_{jk-1} = \{o\}$. Recall that $\sigma_i^- \leqslant \sigma_{k-1}^- < \sigma_i^+ \leqslant \sigma_{k-1}^+$. Putting these facts together, we can deduce that $\sigma_j^- < \sigma_i^- \leqslant \sigma_{k-1}^- < \sigma_j^+ < \sigma_i^+ \leqslant \sigma_{k-1}^+$. Moreover, we know that $\sigma_j = \sigma_i = \sigma_{k-1} = d$ and $\sigma_k = ((\sigma_i^+ + \sigma_j^+)/2, (\sigma_i^+ + \sigma_j^+)/2 + d)$. Hence, $\sigma_j^- < \sigma_i^- \leqslant \sigma_{k-1}^- < \sigma_j^+ < \sigma_k^- < \sigma_i^+ \leqslant \sigma_{k-1}^+ < \sigma_k^+$. We deduce that $\sigma_{k-1}\{o\}\sigma_k$ and hence, $\sigma_{k-1}\ C_{k-1k}\ \sigma_k$. Let $l \in 1, \dots, k-2$. Examine all possible cases concerning the constraint C_{lk}. We know that $C_{lk} \subseteq T$, hence, we must consider four possible cases.

– $C_{lk} = \{eq\}$. From the $\circ$-closure of $\mathcal{N}$, we deduce that $C_{lk-1} \subseteq C_{lk} \circ C_{kk-1}$. Hence, $C_{lk-1} \subseteq \{eq\} \circ \{oi\}$. Since $\{eq\} \circ \{oi\} = \{oi\}$, we have $C_{lk-1} = \{oi\}$. Consequently, $\sigma_l'^+ > \sigma_{k-1}'^+$ and hence, $l > k-1$. This is a contradiction.

– $C_{lk} = \{m\}$. By hypothesis, $E_k^m = \emptyset$. This is a contradiction.

– $C_{lk} = \{b\}$. From the $\circ$-closure of $\mathcal{N}$, $C_{lk-1} \subseteq C_{lk} \circ C_{kk-1}$. Hence, $C_{lk-1} \subseteq \{b\} \circ \{oi\}$. Moreover, $\{b\} \circ \{oi\} = \{b, o, m, d, s\}$ and $C_{lk-1} \subseteq T$. Consequently, $C_{lk-1} \subseteq \{b, o, m\}$. Consider the two following cases:
  - $C_{lk-1} = \{o\}$. We deduce that $l \in (E_k^b \cap E_{k-1}^o)$. By definition of j, we have $l \leqslant j$. Hence, $C_{lj} \subseteq T$. Moreover, $\sigma_l\ C_{lj}\ \sigma_j$. Hence, $\sigma_l^- < \sigma_j^-$ and $\sigma_l^+ < \sigma_j^+$. Since $\sigma_j^+ < \sigma_k^-$, we can assert that $\sigma_l\ \{b\}\ \sigma_k$. Consequently, $\sigma_l\ C_{lk}\ \sigma_k$.
  - $C_{lk-1} = \{b, m\}$. As the property $P(k-1)$ is true, we have $\sigma_l\ C_{lk-1}\ \sigma_{k-1}$. Hence, $\sigma_l^+ \leqslant \sigma_{k-1}^-$. Moreover, $\sigma_{k-1}^- < \sigma_k^-$. Hence, we can assert that $\sigma_l^+ < \sigma_k^-$. Hence, $\sigma_l\ \{b\}\ \sigma_k$. Consequently, $\sigma_l\ C_{lk}\ \sigma_k$.

– $C_{lk} = \{o\}$. From the $\circ$-closure of $\mathcal{N}$, $C_{lk-1} \subseteq C_{lk} \circ C_{kk-1}$. Hence, $C_{lk-1} \subseteq \{o\} \circ \{oi\}$. Moreover, $\{o\} \circ \{oi\} = \{o, oi, d, s, f, di, si, fi, eq\}$ and $C_{lk-1} \subseteq S$. We deduce that $C_{lk-1} \subseteq \{o, oi, eq\}$. The case $C_{lk-1} = \{oi\}$ is not possible since $l < k-1$. We must still examine two cases:

- $C_{lk-1} = \{eq\}$. As $P(k-1)$ is true, we have $\sigma_l\ C_{lk-1}\ \sigma_{k-1}$. Hence, $\sigma_l = \sigma_{k-1}$. Moreover, we know that $\sigma_{k-1}\ \{o\}\ \sigma_k$. We deduce that $\sigma_l\ C_{lk}\ \sigma_k$.
- $C_{lk-1} = \{o\}$. As the property $P(k-1)$ is true, we have $\sigma_l\ C_{lk-1}\ \sigma_{k-1}$. Hence, $\sigma_l^- < \sigma_{k-1}^- < \sigma_l^+ < \sigma_{k-1}^+$. Moreover, we know that $\sigma_{k-1}^- < \sigma_k^- < \sigma_{k-1}^+ < \sigma_k^+$. Consequently, $\sigma_l^- < \sigma_k^-$ and $\sigma_l^+ < \sigma_k^+$. Moreover, recall that the integer i is defined by $i = \min E_o^k$ and $\sigma_k^- < \sigma_i^+$. From the fact that $C_{lk} = \{o\}$ we have $l \in E_o^k$. Hence, $i \leqslant l$. Consequently, $C_{il} \subseteq T$ and, as property $P(k-1)$ is true, we can assert that $\sigma_i^+ \leqslant \sigma_l^+$. Putting everything together, we get the fact that $\sigma_k^- < \sigma_l^+$ and hence, $\sigma_l^- < \sigma_k^- < \sigma_l^+ < \sigma_k^+$. We conclude that σ_l et σ_k satisfy the relation $\{o\}$. $\square$

We can now state the main result of this section.

Theorem 4. *Let $\mathcal{N} = (V, C)$ be a network on $\mathcal{INDU}$ whose constraints are atomic relations defined from* $\mathsf{INDU}^=$. *If $\mathcal{N}$ is closed by $\diamond$ and does not contain the empty relation as a constraint then $\mathcal{N}$ is consistent.*

Proof. Let $\mathcal{N} = (V, C)$ be an atomic constraint network of $\mathsf{INDU}^=$. Let us suppose that $\mathcal{N}$ is $\diamond$-closed and does not contain the empty relation. $\mathcal{N}^{\mathsf{IA}}$ is a $\circ$-closed network (Proposition 14) and moreover, we cannot have the empty relation as constraint. We can notice that the basic relations of $\mathcal{N}^{\mathsf{IA}}$ belong to the relation $S = \{eq, m, mi, b, bi, o, oi\}$. Hence, $\mathcal{N}^{\mathsf{IA}}$ admits a consistent instantiation σ which assigns to the variables intervals with common duration. σ is a consistent instantiation of $\mathcal{N}$. $\square$

Hence, the $\diamond$-closure method is a complete method for the atomic networks on $\mathsf{INDU}^=$.

9. Conclusions

The $\mathcal{INDU}$ calculus lacks many of the nice properties of Allen's calculus: its (weak) composition table does not define a relation algebra. Consistency does not imply 3-consistency, and neither does 3-consistency imply consistency, even for atomic networks: some four node networks are 3-consistent but not consistent. In spite of these negative results, we are able to characterize interesting tractable subsets of relations. To this end, we use both the syntactic approach (Horn classes) and the geometrical approach (convexity and pre-convexity). While the two methods yield the same class in Allen's case, they provide us with two separate tractable subsets in the case of $\mathcal{INDU}$. Following the geometrical approach, we define the set of pre-convex relations and prove that its consistency problem is NP-complete (see also Fig. 5). We then characterize two subsets of pre-convex relations: one is the subset of strongly pre-convex relations, which is tractable (for reasons pertaining to the syntactic properties of its relations), but for which consistency cannot be decided by the $\diamond$-closure method (the usual path-consistency method which uses the

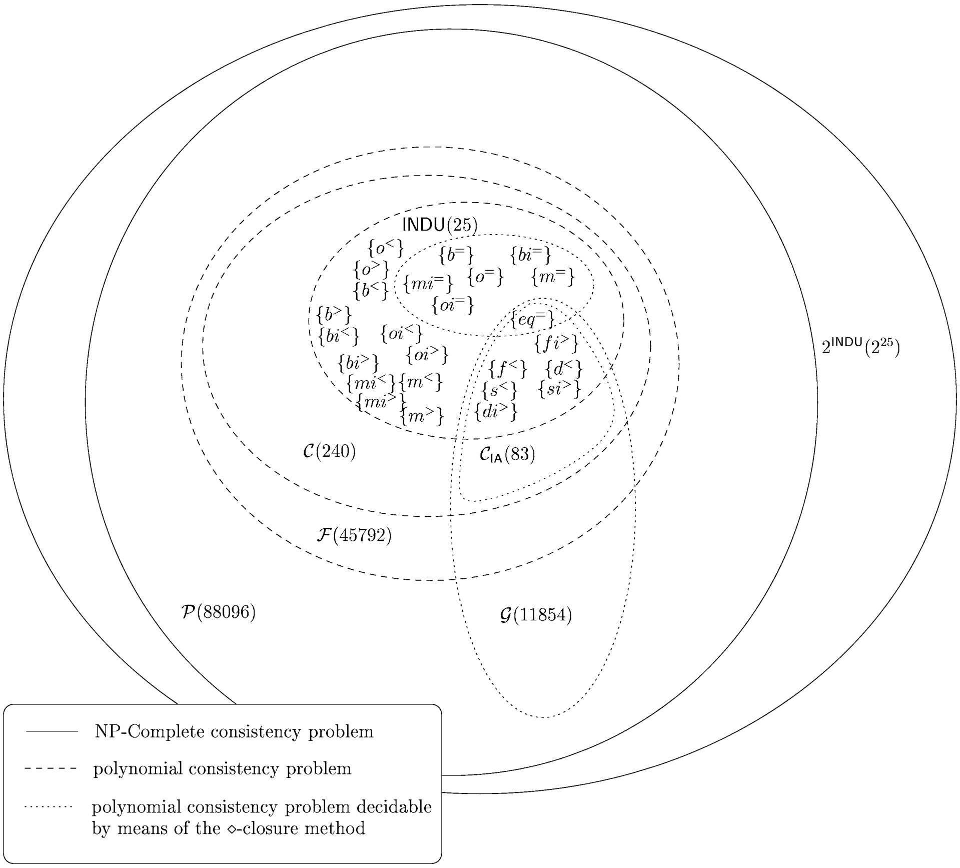

Fig. 5. Recapitulatory of the complexity results.

weak composition operation). The other (incomparable) subclass is tractable and its consistency problem can be solved by the $\diamond$-closure method. In the general case, this method is not complete for the $\mathcal{INDU}$ atomic networks. Despite this, we prove that the $\diamond$-closure method is also complete for the set of atomic relations of $\mathcal{INDU}$ implying that the intervals have the same duration. This paper constitutes a first fully successful exploration of the complexity properties of the $\mathcal{INDU}$ calculus.

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